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Struggles and Agonies in Shaping a Cultural-Historical Conception of Mathematics

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Abstract This chapter is a reflection on the editors' question: Where is the Math in your Mathematics Education Research? The chapter is about the lengthy journey and the agonizing problems I encountered while trying to articulate a cultural- historical conception of mathematics that came to constitute a driving force in understanding the teaching and learning of mathematics in my work. My journey started in the mid-1990s when the Platonists and subjectivist constructivist conceptions of mathematics were on the centre stage of mathematics education. The first part of the chapter sketches the painful efforts to detach myself from the Platonist universalist view of mathematics, from the subjectivist view of constructivism, and from the naïve view of cultural relativism. The second part deals with a reconceptualization of mathematics and learning.

Keywords Cultural-historical activity theory · Cultural conception of mathematics · Theory of objectification · Knowing and learning

1. The Question of Culture and Mathematics

If there is one thing that has not been lacking in philosophy, it is conceptions about mathematics. Two of the most important conceptions are the Platonic and the developmental. In the Platonic conception, mathematics is considered to deal with objects or things in themselves that exist independently of human experience (Bernays, 1935). In the developmental version, mathematics is considered to evolve. Its development is often seen as split into two explanatory spheres: a rational teleological one—the *internal* history—and a non-rational one—the *external* history that deals with the context of discovery and development (see, e.g., Glas, 1993). The internal history is that which accounts for the development of mathematics and mathematical thought, while the external history (the history of society) is taken simply as complementary to the first. As Lakatos famously put it, "*internal history is primary, external history only secondary*" (1978, p. 118; emphasis in the original). In both cases, mathematics is seen as essentially insulated from the vicissitudes of society and culture.

None of these versions of mathematics seemed to respond to the theoretical lines of the research program I began to build in the 1990s—one which would bring for- ward an explicit link among culture, history, and mathematics knowing and learning. If, at that time, sociocultural theories in mathematics education were relatively successful in countering the individualist stance of constructivist theories, in showing for instance the crucial role of language and material culture in teaching and learning (Bartolini Bussi & Mariotti, 1999; Boero et al., 1997; Lerman, 1996; Sfard, 2000), a clearly articulated concept of mathematical knowledge as deeply entangled in culture and history was still missing.

To my dismay, anthropological and sociological studies were of little help, as I found out that, unfortunately, scholars of anthropology and sociology generally avoided messing with mathematics (see, e.g., Berger & Luckmann, 1967; Mannheim, 1955). In philosophy, the case was pretty much the same. In his *Origin of Geometry*, Edmund Husserl (1939) tried to understand the status of scientific and mathematical objects by reflecting on the movement of concrete consciousness (the starting point of a transcendental phenomenology) from material objects to their transempirical essences. Considering mathematics as an example of pure sciences of essences, he strove to understand the foundational acts of meaning in their relationship with the allegedly a-cultural universal truth of mathematical objects. His account, however, remained helplessly subjugated to the universalism that remains incompatible with a cultural-historical account of knowledge and knowing. In Radford (2006), I argued that Husserl came to intuit that meaning and conceptual objects coexist with culture, but in no case could he conceive of the latter as consubstantial of the former. As Derrida notes,

> Besides all the characteristics that it has in common with other cultural formations, [for Husserl] science claims an essential privilege: it does not permit itself to be enclosed in any historically determined culture as such, for it has the universal validity of *truth*. As a cultural form which is not proper to any de facto culture, the idea of science is the index of pure culture in general [. . .] Science is the idea of what, from the first moment of its production, must be true always and for everyone, beyond every given cultural area. (Derrida, in Husserl, 1989, p. 58)

So, my question remained unanswered: How can we link, in an organic manner, culture and the production of mathematical knowledge? I considered this question to be the first step towards the creation of a cultural-historical perspective on teach- ing and learning. Indeed, since all learning is about learning *something* (in our case mathematics), the only way is to start by clarifying the cultural-historical nature of mathematics.

2. Mathematics Creation in Ancient China and Ancient Greece

In 1996, I came across Lizcano's (1993) book on mathematics creation in ancient China and ancient Greece. Rather than assuming the classical universalist view of mathematics, Lizcano was exploring through the mathematics of these ancient civilizations, how mathematics "emerge *contaminated* by the collective imaginary meanings that are latent in the reason of each epoch and each culture" (p. 13). He asked:

> How does each society construct the bar that divides — and links — the possible and the impossible, the real and the imaginary, the thinkable, the true and the false? How does the social imagination of space influence the location of mathematical objects? (p. 14)

To do so, he resorted to Castoriadis's (1975) concept of collective imaginary—a symbolic dimension that people of a culture share and that is based on a specific ontology of the world. Lizcano was particularly interested in investigating how the Chinese mathematicians were able to come up with a view of negative numbers, something unthinkable for the Greek mathematicians of the Classical period. His explanation rests on their different ontologies. The Greeks made their ontology of mathematics revolve around the *dichotomy* of *being/non-being*. From this ontological principle derives the principle of the excluded third that plays a fundamental role in asserting truths about things—as Euclid did. By contrast, the Chinese made their mathematics revolve around the *complementarity* of *yin-yang*. *Yin-yang* works as a symbolic wellspring of opposing forces that gave meaning to ancient Chinese everyday practices as diverse as divinatory or culinary ones. Lizcano found that this symbolic wellspring of opposing forces was at work in the ancient Chinese conception of mathematics. It also gave meaning to the space of a board and the red and black chopsticks used in the *zheng fu* method to solve equations. In this problem-solving method, the red chopsticks relate to positive numbers, while the black chop- sticks relate to what is usually translated as negative numbers. Negative numbers are thinkable here as they emerge charged with the idea of *opposition* in the *ying-yang* sense. I read this book at ICME 1996 during the hot evenings of a Sevilla summer and ended up writing a review. In my review, I pointed out that:

> The symbolic complex of *yin/yang* appears not only as a punctual oppositional organizer, as would be the case in instances such as masculine/feminine, open/closed, etc., but as an organizer of discursive fields as complex as poetry. (Radford, 1996, p. 403)

Lizcano's book was subjected to relentless criticism. For example, Echeverría (1995) saw in Lizcano's book a concrete example of that tiresome and unpleasant posture that universalist theorists call cultural relativism. The reception of Lizcano's work is a sample of the difficulties that had to be overcome in sociology and other areas to conceive the idea that mathematics is anchored in its own culture. D'Ambrosio (1993) and the ethnomathematicians were also very influential in gathering a great deal of data that were showing the role of culture in mathematics cognition. In her work with aboriginal communities in Australia, Owens (2001, p. 157) concluded that mathematics was subjected to taboos much like other cultural products: "like food taboos," she said, "mathematics is not free of other, very significant aspects of culture."

Conceptions about mathematics were certainly changing.

However, neither Lizcano's book nor ethnomathematics research solved my problem entirely. In Lizcano's account, mathematics appears in the light of a sub- lime symbolic order (*being/non-being, yin-yang*), but still detached from the tumultuous world of politics and concrete life. A link was still missing. There was still a long way to go to achieve a cultural-historical operational definition of mathematics in the investigation of its teaching and learning. Thus, while continuing my research on the history and epistemology of mathematics, I turned to Vygotsky, Leont'ev, Davydov, and to dialectical materialism. It was about 20 years after ICME 1996 that, trying to overcome the narrow scope of the matrix of symbolic cultural orders, my search for a cultural-historical articulation of mathematics took shape. In the next sections I sketch some aspects of this search.

3. Knowledge

During the same decade that Lizcano published his book and the ethnomathematicians continued making progress, constructivists were busy articulating their view of learning. Following Piaget, as interpreted by von Glasersfeld (1995), constructivists came up with a subjectivist conception of knowledge. They argued that knowledge is what each individual constructs from their actions and interaction with others. In their account, culture and society appear as a mere set of stimuli. In the constructivist account, knowledge is a *mental subjective entity* that originates from the individual's own experiences. Later on, pressed by sociocultural theorists, con- structivists added the prefix "socio-", but the ensuing socio-constructivism had to remain truthful to its main idea of knowledge as personal construct (Cobb et al., 1997), the result being a very poor concept of the social, reduced to the mere inter- action between people, or, at best, the facilitative grooming individuals need “to become more fully socialized and intellectually engaged” (Martin, 2004, p. 197).

In the field of philosophy, Ilyenkov (1977) tried to understand the cultural and historical underpinnings of knowledge — or more precisely, what he termed *idealities*. Drawing on Spinoza, Hegel, and Marx, knowledge in his account is conceptualized as a mode of theoretical reproduction of reality. I found Ilyenkov’s work and Bakhurst’s (1988, 1991) analysis of it very inspiring. Reflecting on Ilyenkov’s and Bakhurst’s work allowed me to move beyond the individualist stance of knowledge of constructivism, as well as the behaviourist conception of knowledge as conditioned or imitative responses to stimuli. I was now able to see knowledge from a broader stance where it appears not as a mental entity but as a cultural-historical one. Although I felt that I was making some progress, a direct transposition of Ilyenkov’s ideas to mathematics education was not possible. It was clear, however, that thematizing mathematics along the lines of Ilyenkov’s philosophy could help me overcome the limits of framing mathematics within the confines of a sublime symbolic order, as Lizcano did. I could now see that there was a possibility to link mathematics and its production to the tumultuous world of politics and concrete life. But, as before, there was still a long way to go to achieve a cultural-historical operational definition of mathematics in the investigation of its teaching and learning. I read Marx again and embarked on a reading of Hegel and Spinoza. The technical philosophical jargon of Hegel and Spinoza easily discourages the novice reader, but I persisted. My lengthy and difficult philosophical adventure was helpful. For one thing, I understood much better Vygotsky’s, Leont’ev’s and Davydov’s work. I realized that, to understand them, we have to read them as what they were: dialectical materialist thinkers (see, e.g., Radford, 2021a). Concepts such as the zone of proximal development, consciousness, internalization, and activity appeared in a new light.

4. Towards a Cultural-Historical Conception of Mathematics

In the educational cultural-historical perspective I wanted to develop, two things were becoming clearer. From a phylogenetic viewpoint, knowledge should be articulated as a dialectical dynamic entity, always changing in its intertwining with concrete practices. From an ontogenetic viewpoint, knowledge should be understood as an entity already present in our culture. While the phylogenetic viewpoint emphasizes the cultural situatedness of knowledge and its historical nature, the ontogenetic viewpoint emphasizes the effect knowledge has on us: when we are born, what we find in front of us is not a mere complex of material objects, but also a complex of cultural-historical ideas that affects us as we grow.

I realized that such an articulation of knowledge was leading me to a reconceptualization of the students. Indeed, while in constructivism and many other theories, the student is seen as the origin of knowledge, meaning, and intentionality—that is to say, as a founding subject, a subject that posits the world— in my account, the student was appearing rather as a subject that "comes into the world" (Kemp, 1973). I was led to talk about this idea in an interview in Brazil (Moretti et al., 2018), when I was asked about the concept of knowledge in what was becoming the theory of objectification. In this interview, I suggested to imagine two scenarios: the first is a rural community that has produced ideas about time, space, numbers, how to sow the soil, etc. The second is a community based on capitalist forms of mercantile production as in a contemporary European or North American country. Now imagine two babies born at the same time in each one of these communities. Both babies will find in front of them a complex of different cultural ideas that will affect them as they grow. For instance, a baby born in the Mi'kmaw community that Borden (2013) describes will grow counting things in a different way from a New York baby. While the latter will grow understanding that the smell, colour, and sizes of counted things do not matter, the former will not, for "in Mi'kmaq *what* one counts determines how one counts" (p. 6).

Hegel was very helpful to give body to the idea of knowledge I was after. In particular, he was helpful in my attempt to theorize knowledge as complex of cultural-historical ideas that function as a general disposition (a *potentiality*) to act, understand, interpret, and transform the world. Let me explain.

In Hegel's philosophy and its ensuing dialectical materialism, ideas are always manifested through some sort of action. Let us take the example of numbers. We count by pointing to the counted objects, or by eidetically thinking of them, or through a combinatorial formula, etc. The point is that ideas are not merely general ideas (potentialities or dispositions), but are always *manifested* in one tangible way or another. Following Aristotle, to thematize the immanence of action in its potential form, Hegel distinguished between *potentiality* (dunamis) and *actuality* (energeia). But in contradistinction to the Greek philosopher, Hegel conceived of these categories as part of a dialectic system where the potential is *revealed* in the act(uality) of its appearance — like the concrete sound reveals the potential sounds of a piano or a concrete action of solidarity reveals our care for someone. In Hegel's account, these two categories, that are theoretically distinguished, *come together in action*, in "actual reality" (Hyppolite, 1974, p. 292). Here, Hegel refutes Kant and his distinction between the *thing-in-itself* (e.g., the triangle) and its phenomenological appearance (the drawn triangle). By arguing that the *thing-in-itself* becomes embedded, embodied, or incarnated in its manifestation or actualization, Hegel negates the abstract-concrete dichotomic conception of Kant's rationalism as well as the general-particular dichotomy of the empiricist philosophers.

To understand this point better, which, as we shall see, plays a fundamental role in my conception of mathematical knowledge, it is worthy to see with some detail Hegel's idea of "actual reality" or effective reality. Hegel's expression is *Wirklichkeit*. In terms of our discussion, effective reality is what a culture produces for itself from its always evolving potentialities and under the action of its various concrete activities. There is a dialectical movement between potentiality and actuality that contains contingency and an increasing wealth of new possibilities. It is indeed possibility as part of the *Wirklichkeit* that impedes effective reality to be reduced to the empirical world. The very texture of effective reality is made up of the dialectical units of potentialities and actualities that, in their movement and with all the contradictions this movement entails, always open spaces for new possibilities of action and thought.

How can we conceive of knowledge in this context? Often, as mentioned above, mathematics is seen as constituted of transcendental objects (this is, for example, the Platonic view) or abstract objects (this is the Aristotelian view). In Radford (2021b), I suggested to consider knowledge along the lines of Hegel's concept of potentiality—knowledge as comprised of cultural-historical *ways of thinking*. In this view, mathematics is not really about triangles or numbers *per se*, but about how we *think* about things (e.g., forms, quantities, motion, etc.). It is about how we *deal* with things (assert truths, represent things, draw conclusions, generalize, and so on). More precisely, mathematics is a dialectical system of forms of thinking, action, and reflection constituted historically and culturally out of material, embodied, and sensible collective labour.

In this view, knowledge is the ideational counterpart of cultural practices — or, to say it with a Spinozist accent, as Ilyenkov did, a mode of theoretical reproduction of effective reality. Mathematical knowledge is not an exception. This is the theoretical line that I followed in my investigation of ancient Greek mathematics (see Chap. 8 in Radford, 2021b), where I tried to put into evidence three forms of Greek mathematical thinking, each one related to its own practices (the Athenian banking practices focused on quantifications and numerical calculations in business and financial matters, the geometric practices of surveyors and architects, and the theorematic practice of aristocratic society). These forms of mathematical thinking do not merely live side by side. I tried to show that, in fact, they reflect the contradictions of their society, in particular, the dialectical contradictions arising from the ancient Greek basic antagonism between slave and citizen. Reading these contradictions dialectically, one can see that each of these forms of mathematical thinking *negates* the others. In dialectical materialism, this negation does not amount to the mere separation or opposition of things. Actually, it is the opposite: negation means a *distinction* that *affects* the negating and the negated by mutually affecting both in the configuration they acquire.

If we bear in mind the idea of *Wirklichkeit,* the effective reality where idealities and their manifestations or concretions come together, we can understand mathematics as an entity that is at the same time ideal, sensible, and material. Mathematics *appears* much in the same way as music when an orchestra plays, say, a symphony. Like music, mathematics is something that appears as students and teachers engage in classroom activity (Radford, 2019). What appears in the mathematics classroom is visual, tactile, olfactory, aural, material, artefactual, gestural, and kinesthetic, and, being all of that, becomes an object of consciousness and thought. School mathematics, in this materialist and phenomenological line of thought, is what is made sensible through the teachers' and students' classroom activity.

5. Learning

The articulation of the dialectical materialist concept of mathematics sketched in the previous section was not an end in itself. As a mathematics educator, I am mostly interested in teaching and learning. However, to understand teaching and learning I needed first to come up with a cultural-historical concept of mathematics. The next problem was to link this concept to learning.

Sociocultural theories have resorted to a series of concepts to understand learning. One of them is the concept of *participation*, which was developed by Rogoff (1990) and Lave and Wenger (1991), among others. Rogoff, for example, conceives of learning as *apprenticeship* in a context of guided participation. In the footsteps of Rogoff, and drawing on ethnographic research on craft apprenticeship among Vai and Gola tailors in Liberia, Lave and Wenger explored the concept of learning through the construct of *legitimate peripheral participation*.

The theoretical construct of participation with its focus on how individuals learn to do things in a culture is certainly interesting. It is inscribed within the larger project of enculturation. However, I wanted to see learning differently. Learning should be more than entering into a culture. I wanted to emphasize with greater force the agentic dimension that underpins learning.

To some extent, this agentic dimension was also missed by the Vygotskian concept of internalization. Contemporary Vygotskian researchers have called attention to this point. For instance, González Rey argues that in the investigations of internalization, "subjectivity and the subject [became] mere epiphenomena of discursive, semiotic, and linguistic practices" (2011, p. 36). Stetsenko (2020) has claimed recently that the theorizing of agency and subjectivity is "one of today's major challenges in cultural-historical activity theory" (p. 5).

If the constructs of participation and internalization do not seem suitable to theorize learning as I wanted to do it, to what concept can I resort? In seeking an answer, I followed Vygotsky, who, when talking about learning, contemplated the concept of *consciousness.*[1] Vygotsky understood consciousness not in a metaphysical sense, but in a materialist one; that is, as *becoming conscious* or *aware* of something in a transformative sense: by becoming conscious of something we transform and, at the same time, become transformed by it.[2]

I found in Vygotsky and Leont'ev, as well as in the work of another dialectical materialist thinker, the Brazilian educator Paulo Freire (2005), the support to articulate what I was looking for: learning as a transforming event intertwined with the aesthetic, cognitive, and political flux of social experience against the always con- tested background of culture and history.

We can now try to put all the pieces together. Considering knowledge as a dynamic system of forms of thinking constituted historically and culturally, learning can be thematized as the students' *encounter* with these forms of thinking. At the outset, these forms of thinking escape to the students' consciousness. They are there, in the students' culture, in front of them, so to speak, but they are not yet noticed. We can use the verb "to object" to describe this situation. To the extent that these forms of thinking are there, still unnoticed, they *object* the students (as, if the metaphor is allowed, a chair objects us). In Radford (2002), I used the verb "to object" in this sense, prompted by a letter that the painter Vincent van Gogh wrote to his brother, Theo. In this letter Vincent writes: "Theo, what a great thing tone and colour are […] M[auve] has taught me to see so many things that I used not to see and one day I shall try to tell you what he has told me" (1997, p. 114).

In their cultural-historical nineteenth-century artistic articulation, tone and colour, were there, yet something had to happen for van Gogh to *see* them. This process that allowed van Gogh to learn about tone and colour; that is, to become conscious of them, is what I have termed *objectification*, which etymologically means a process aimed at bringing something in front of someone's attention or view.

Processes of objectification are the active, embodied, discursive, symbolic, and material processes through which the students actively encounter, notice, and become critically acquainted with culturally and historically constituted systems of thinking, reflection, and action. In this encounter, the students are faced with the alien, the Other. Processes of objectification include those acts of meaningfully noticing something that reveals to our consciousness through our bodily, sensory, and artefactual semiotic deeds.

Seeing learning through the lenses of processes of objectification means seeing it as dialectics; that is, as *movement*: learning as the dance of embodied consciousness, this back-and-forth movement where knowledge appears and, by appearing, creates a contradiction. It disturbs what we know, what we think, and what we believe. By disturbing us, knowledge invites us to make sense of it; it invites us to think differently, to transform it. And to do so, we are invited to stretch our current possibilities, to use our imagination and our creativity.

A short example may help illustrate these ideas. The example comes from my current research with daycare students. Figure 12.1 shows two students learning about numbers. The educator gives Thomas yellow blocks and James blue blocks. She asks them to make a tower; then, she asks who has *more* blocks.

Thomas answers that he does. The educator asks the students to count the number of blocks in each tower, and helps the children count. However, Thomas still argues that he has more blocks. She changes strategy: she suggests that they place the towers side by side, hoping that the perceptual strategy would lead them to answer the question. The perceptual strategy does not help as expected: each child argues that he has more blocks than the other.

Fig. 1 Thomas (left) and James (right) dealing with a comparison of numbers

The students are in the process of learning: they are encountering a cultural-historical form of mathematically thinking about numbers and the cultural procedures that allow one to assert that a collection has more elements than another. Behind the apparent transparent comparison of quantities, rests a precise complex meaning of numbers and a cultural way of ascertaining truth. Like in the case of van Gogh, the cultural way of thinking is going to be disclosed through the dialectics between potentiality and actuality, as mathematical knowledge appears progressively and sensuously to the students' consciousness in effective reality, *Wirklichkeit*. It is important to emphasize the dialectical nature of the encounter with knowledge. This encounter is such that the students are actively and creatively engaged in its actualization, so that what appears bears the mark of both, culture and the subjectivities of the students and teachers. This is why the appearance of knowledge is always new, always different. Its appearance is the dialectical symbiosis of two elements—subjects and culture—that become *one*, carrying in itself the contradictions and negations of the various voices and perspectives that make mathematics what it is—not a set of disembodied truths, but the agentic movement of actuality.

6. By Way of Conclusion

In this chapter, I wanted to share some of my own struggles and agonies in trying to come up with a theoretical concept of mathematics and learning from a sociocultural perspective. My interest in this problem was shaped by some key experiences in my life. This may be true in general: the most important interests we all develop have roots in the context where we live our life. I was born and grew up in a small country—Guatemala—where pre-Colombian and mixed cultures issued from European colonization come together in a dynamic intertwined with history, power, race, and politics. Growing up in such a context, I was confronted every day by a variety of ways to think about the world. When I went to France to study mathematics and the didactic of mathematics, I found myself living in a society with different values, institutions, and understandings of the world. The same occurred when, some years later, I moved back to Guatemala and then to Canada. The sense that emerged from these experiences was that there was something deeply wrong with the European Enlightenment view of a universal reason, and a universal mathematics (Radford, 2017). Of course, one of the main weapons of postmodernism has been its attack against the grand narratives of modernity. However, there is a long way to go from this postmodernist move to a clear articulation of the relationship between knowledge and culture. As many critics have argued, postmodernism has been unable to decisively move beyond the individualist conceptions of modernism (Cole & Hill, 2002) and the ensuing views of society and culture (Eagleton, 1996). This is most clearly visible in post-modernist ethics, both in their moderate versions (e.g., Bauman, 1993) as in the radical ones (e.g., Levinas, 1979).

The re-interpretation of Hegel in the work of Ilyenkov, Marx, Bakhurst, Vygotsky, Freire, and others provided me with what I needed to articulate a perspective that is neither structuralist, nor modernist, nor postmodernist in order to offer a theoretical answer to the riddle of the link between knowledge and culture—a link that has incessantly haunted sociocultural theoreticians. The theoretical answer rests on a conception where knowledge is not considered the psychological individualist con- struct of empiricist and rationalist epistemologies but a cultural-historical entity. Knowledge is conceived of as a system of ways of thinking, reflecting, languaging, acting, and doing—a system logged in the culture and that is produced, and continuously transformed, in *sensible and material, human activity*.

In this view, mathematics is simultaneously ideal and material, and appears in the classroom through the mediation of teaching-learning activity. It appears as a visual, tactile, olfactory, aural, material, artefactual, gestural, symbolic, and kinesthetic "palpable" entity (Radford, 2009). It appears through the teacher's and students' interactions, symbols, diagrams, gestures, words, etc.

One might ask: What practical difference does all this make to teaching and research? Here is a partial twofold answer.

First, the conceptualization of mathematics that I have just outlined leads to a conception of learning as encounter—an encounter with cultural-historical knowledge. In doing so, in practical terms, we move away from the Kantian-Piagetian individualist conceptualizations that became prominent when mathematics education tried to overcome the limits of direct teaching after World War II.

We are now in a position to rethink learning—to rethink it as a *transformative process*. Seeing the encounter in dialectical terms—that is, in terms of a dialectics between what is to be encountered (knowledge) and those encountering it (learners)—opens a possibility to understand teachers and students as subjectivities in the making. Teachers and students appear entangled with knowledge, not only understanding and trans- forming it, but also disturbing it, subverting it. This is what Thomas does when he keeps claiming that he has more blocks than James, reminding us that there are many ways to think about numbers and collections. *At the same time*, by being con- fronted by other ways of thinking about numbers (in this case, a Western cultural way of thinking), Thomas is *challenged*. Education gives him the precious gift of overcoming the solipsistic stance in which we could be confined were it not for the presence of what we are not.

The second part of my answer has to do with the claim made above that mathematics appears in the classroom through the mediation of teaching-learning activity. Its appearance is hence correlated to the *manner* in which teaching-learning activity unfolds. This is why alienating teaching-learning activities unavoidably leads to alienating learning, as in direct teaching, where teachers and students are disempowered (Radford, 2012). The practical implications here are about imagining and implementing the kind of teaching and learning activity that would be conducive to an emancipative *praxis*. In my recent work, this praxis is termed *joint labour* (2021b). Although teachers do not necessarily do the same thing as the students, teachers and students work *together* to make mathematics appear and appreciate it critically. Brazilian educator Cristiane Nery contends that, in her research, under- standing knowledge as presented here, and teaching and learning as joint labour, indigenous teachers became increasingly "aware of the different forms of knowledge developed historically and culturally" (2023, p. 10), which proved central "to raise awareness of indigenous socio-cultural mathematical knowledge" (p. 10).

So, to the reviewer's question, "Do I just go back to my classroom or research unit and carry on as before, but now with a deeper understanding of what I'm doing," I would answer no. As a teacher or as a mathematics educator, I can no longer ignore my unavoidable participation in the political and ideological dimensions of education that will always be present when I teach. I have choices to make. This is why teaching, we come to realize, is ethical. The possibility of ethics, Vygotsky argued, rests on the "full of unrealized possibilities" by which we are confronted every minute (Vygotsky, 2003, p. 76). The question now moves to envisioning ethics as a liberating force that can help us come to grips with the contemporary historical, political, and economic sources and structures of oppression, violence, and inequality.

Endnotes

[1] See, e.g., Vygotsky's *Preface* to Koffka's *Foundations of the Grow of the Mind*. The *Preface* was published in Chap. 14 of Vygotsky (1997).

[2] Leont'ev (1978) explains the concept of consciousness as follows: "consciousness is not a manifestation of some kind of mystical capability" (p. 19). "Consciousness is not thought plus perception plus memory plus ability. Or even all of these processes taken together plus emotional experience. Consciousness must be understood … not only as knowing but also as relations, as direction" (p. 145), the subjective relations and directions that orient us in the world.

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