

ALGEBRAIC THINKING AND THE GENERALIZATION OF PATTERNS: A SEMIOTIC PERSPECTIVE

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The most important operation of the mind is that of generalization.
C. S. Peirce, *Collected Papers* 1.82.

Introduction

Several years ago, I had the opportunity to conduct longitudinal research in four Junior High-School classes about the teaching and learning of algebra. The timing was just perfect: the previous year, i.e. 1997, the Ontario Ministry of Education released a new Mathematics Curriculum based on a new type of assessment, the enlargement and reorientation of knowledge content and the rigorous description of the expected learning. To say the least, teachers were worried about the new high expectations. The time was just ripe for collaboration. There was a clear sense in the educational community that, in order to implement the new curriculum, we all had a lot to learn from each other. For me, working with three or four teachers every year for six years and following the same students in the classroom as they moved through Junior and Senior High School constituted a marvellous opportunity.

We designed a flexible teaching-researching agenda committed to meeting two main goals: First, we wanted the students to learn the algebraic concepts stipulated by the Curriculum. This was a practical concern framed by the aforementioned political educational context. Second, we wanted to deepen our understanding of the emergence and development of students' algebraic thinking, the difficulties that the students encounter as they engage in the practice of algebra and the possible ways to overcome them. The longitudinal research was characterized by a continuous loop: (1) classroom activity design → (2) classroom activity implementation → (3) data interpretation → (4) theory generation → (1) classroom activity design → (2) ... Our longitudinal research was informed by the wealth of research previously conducted on the transition from arithmetic to algebra. In the early 1980s, Matz (1980) and Kaput and Sims-Knight (1983) investigated some errors associated with symbol use and Kieran (1981) pointed out different concepts associated with the equal sign. Some years later, Filloy and Rojano (1989) put into evidence some key problems that novice students face in solving equations; a bit later Sfard (1991) and Gray and Tall (1994) called attention to the students' difficulties in distinguishing between objects and processes, while Bednarz and Janvier (1996) studied the effects of word problem structure in arithmetic and algebraic reasoning. At about the same time, several researchers showed the limits of X-Y numerical tables in the generalization of patterns (Castro Martínez, 1995; MacGregor & Stacey, 1992, 1995). It was apparent that X-Y tables were emphasizing a formulaic aspect of generality based on trial and error heuristics, hence confining algebraic notations to the status of place holders bearing very limited algebraic meaning.

The research conducted in the 1980s and 1990s –the above sketch of which is obviously incomplete– led to an unavoidable and difficult question asked again and again: that of the exact nature of algebraic thinking. Commenting on the Research Agenda Conference in Algebra (Wagner & Kieran, 1989), held in March 1987 at the University of Georgia, Kieran (1989, p. 163) said: “One of the topics pointed to in the Research Agenda ... as an area sorely in need of

research attention is that of algebraic thinking.” Certainly, since then, the several studies conducted by mathematics educators and historians have made an important contribution to this area (e.g. Arzarello & Robutti, 2001; Boero, 2001; Carraher, Brizuela, & Schliemann, 2000; Høyrup, 2002; Lee, 1996; Lins, 2001; Martzloff, 1997; Puig, 2004; Ursini & Trigueros, 2001). And if we still do not have a sharp and concise definition of algebraic thinking, it may very well be because of the broad scope of algebraic objects (e.g. equations, functions, patterns, ...) and processes (inverting, simplifying, ...) as well the various possible ways of conceiving thinking in general.

It is clear that algebraic thinking is a particular form of reflecting mathematically. But what is it that makes algebraic thinking distinctive? Trying to come up with a working characterization to guide our research, we adopted the following non-exhaustive list of three interrelated elements. The first one deals with a sense of *indeterminacy* that is proper to basic algebraic objects such as unknowns, variables and parameters. It is indeterminacy (as opposed to numerical determinacy) that makes possible e.g. the substitution of one variable or unknown object for another; it does not make sense to substitute “3” by “3”, but it may make sense to substitute one unknown for another under certain conditions. Second, indeterminate objects are handled *analytically*. This is why Vieta and other mathematicians in the 16th century referred to algebra as an *Analytic Art*. Third, that which makes thinking algebraic is also the peculiar *symbolic* mode that it has to *designate* its objects. Indeed, as the German philosopher Immanuel Kant suggested in the 18th century, while the objects of geometry can be represented ostensively, unknowns, variables and other algebraic objects can only be represented *indirectly*, through means of constructions based on signs (see Kant, 1929, p. 579). These signs may be letters, but not necessarily. *Using letters does not amount to doing algebra*. The history of mathematics clearly shows that algebra can also be practiced resorting to other semiotic systems (e.g. coloured tokens moved on a wood tablet, as used by Chinese mathematicians around the 1st century BC and geometric drawings as used by Babylonian scribes in the 17th century BC).

Drawing on the working characterization of algebraic thinking sketched above and the then-emerging Vygotskian perspective in mathematics education (Bartolini Bussi, 1995; Lerman, 1996), we formulated our research problem in semiotic terms. Starting from a broad conception of signs, we wanted to investigate the students’ *use of signs* and processes of *meaning production* in algebra. Naturally, contemporary curricula favour the alphanumeric algebraic symbolism. It was our contention, however, that, ontogenetically speaking, the students’ formation of the corresponding meanings and rules of sign-use were rooted in other semiotic systems that they had already mastered. Since the history of mathematics suggests that, in some cultural traditions, the evolution of some algebraic notations relied heavily on speech (Radford, 2001), we had strong reasons to look to language for the antecedents of the students’ alphanumeric algebraic meanings. The results that we obtained during the first years confirmed our hypothesis, but, as we shall see in a moment, we also came to realize that language was only part of the story.

In this paper, I want to present an overview of some of our results. Although our general goal was to investigate the various aspects of students’ algebraic thinking, as related to the algebraic concepts stipulated by the Ontario Curriculum, for the sake of simplicity, I will focus here on the generalization of patterns only (some results concerning equations can be found in Radford, 2002a, 2002b; Radford and Puig, in press). In Section 1, I suggest making a distinction between *generalization* and (naïve) *induction*. I will claim that, just as not all symbolization is algebraic, not all patterning activity leads to algebraic thinking. I will argue in particular that this is the case

for inductive reasoning (as frequently used by the students), even if the inductive process can be expressed in symbols, such as “ $2n+1$ ”. I will even go further and claim that, among the possible forms of generalization, not all are algebraic in nature (there are some pattern generalizations that are arithmetic but not algebraic, a point that I discuss later in the paper). One practical result that comes out of this is the following. In the use of patterning activities as a route to algebra, we –teachers and educators– have to remain vigilant in order not to confound algebraic generalizations with other forms of dealing with the general; we also have to be equipped with the adequate pedagogical strategies to make the students engage with patterns in an algebraic sense. In Section 2, I discuss the theoretical construct of knowledge objectification, which I use in the subsequent sections to give an account of the students’ sign use and meaning production in classical pattern activities.

1. Towards a Definition of Algebraic Generalization of Patterns

One of the introductory activities to algebraic symbolism that we proposed to Grade 8 students included the classical pattern shown in Figure A.

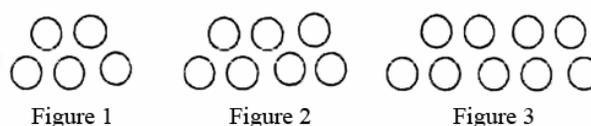


Fig. A. The sequence of figures given to the students in a Grade 8 class.

At the beginning of the activity, the students –who always worked in small groups of two to four– were required to find the number of circles in Figure 10 and in Figure 100. Their strategies fell into two main categories.

In the first one, the heuristic is based on *trial and error*: the students propose simple rules like “times 2 plus 1”, “times 2 plus 2” or “times 2 plus 3” and check their validity on a few cases. The symbolization of the rule may vary. Here is one provided by one of our small groups: “ $n \times 2 (+3)$ ”. When the students of this small group were asked to explain how they found this rule, they said: “We found it by accident.”

In the second one, the students search for a *commonality* in the given figures. Mel, for instance, wrote: “The top line always has one more circle than the number of the figure and the bottom line always has two circles more than the number of the figure”. Mel’s formula was: “ $(n+1) + (n+2) =$ ”

Although both procedures lead to the use of symbolism, the heuristics are incommensurately different. The latter rests on *noticing* certain common features of the given figures and *generalizing* them to the figures that follow in the sequence. In contrast, the former rests on a rule formed by guessing. Rules formed in this way are in fact *hypotheses*. This way of reasoning works on the basis of *probable reasoning* whose conclusion goes beyond what is contained in its premises. In more precise terms, it is a type of *induction* –a type that I will qualify as *naïve* to distinguish it from other more sophisticated types of induction¹. Thus, instead of generalizing something, when resorting to the first procedure, the students merely make an induction and not a generalization².

The comparison of the two aforementioned strategies emphasizes an important distinction between induction and generalization –a difference that is often overlooked and that ends up

calling something generalization while in reality it is simply an induction (Peirce, CP 2. 429). At the same time, it suggests one of the traits that may constitute the core of the generalization of a pattern, namely the capability of noticing something general in the particular, a trait upon which Love (1986), Mason (1996) and others have previously insisted.

Kieran, however, claimed that this trait alone may not be sufficient to characterize the *algebraic* generalization of patterns. She argued that in addition to seeing the general in the particular, “one must also be able to express it algebraically” (Kieran, 1989, p. 165). To understand Kieran’s objection, we should bear in mind that usually, the generalization of patterns as a route to algebra rests on the idea of a natural correspondence between algebraic thinking and generalizing. Kieran took argument against the alleged natural character of this correspondence and contended that to think algebraically is more than thinking about the general. It is to think about the general or the generalized in a way that makes it distinctively algebraic in its *form of reasoning as in its expression*. As she put the matter, “a necessary component [of algebraic generalization] is the use of algebraic symbolism to reason about and to express that generalization.” (Kieran, 1989, p. 165).

I concur with Kieran’s exigency concerning the inclusion of one’s ability to *express* the general. Following a Vygotskian thread to which I shall return in the next section, what I would like to add here is that algebraic generality is made up of different layers –some deeper than others. Furthermore, the scope of the generality that we can attain within a certain layer is interwoven with the *material form* that we use to *reason* and to *express* the general (e.g. the standard alphanumeric algebraic semiotic system, natural language or something else).

In this line of thought, I want to suggest the following definition. Generalizing a pattern *algebraically* rests on the capability of *grasping* a commonality noticed on some elements of a sequence S, being aware that this commonality applies to *all* the terms of S and being able to use it to provide a direct *expression* of whatever term of S.

In other words, the algebraic generalization of a pattern rests on the noticing of a local commonality that is then *generalized* to all the terms of the sequence and that serves as a warrant to build expressions of elements of the sequence that remain beyond the perceptual field. The generalization of the commonality to all the terms is the formation of what, in Aristotelian terminology, is called a *genus*, i.e. that in virtue of which the terms are held together (see e.g. Aristotle’s *Categories*, 2a13-2a18). Direct expression of the terms of the sequence requires the elaboration of a rule –more precisely a *schema* in Kant’s terms (Radford, 2005a). I will come back to this definition later. For the time being, I want to stress two main elements involved in the definition. On the one hand, there is a *phenomenological* element related to the grasping of the generality. On the other hand, there is a *semiotic* element related to the expression through signs of what is noticed in the phenomenological realm. In the next section, I will argue that these two elements are interrelated and that they may be investigated through two theoretical constructs –knowledge objectification and the concomitant semiotic resources to achieve it.

2. Knowledge Objectification

For the novice student, noticing the underlying commonality of the terms of a pattern is not something that happens all of a sudden. On the contrary, it is a gradual process underpinned by a dynamic distinction between the same and the different. Even in a pattern as simple as the previous one (see Figure A), there are several ways to look for what may qualify as the same and the different in the given figures. Thus, talking to his two group-mates, Doug –a Grade 9

student— says: “So, we just add another thing, like that”. At exactly the moment he utters the word “another”, he starts making a sequence of six rhythmic parallel gestures (see *Fig. B*).



Fig. B. Excerpt of Doug’s sequence of rhythmic gestures.

Naturally, the figures all have the *same* shape, but at the same time, they are *different*: that which makes them different, Doug is suggesting, is the last two circles *diagonally* disposed at the end of each figure (see Figure C).



Figure C. Doug emphasizes the last two circles in an attempt to notice a commonality in the terms of the sequence

We see hence that Doug’s grasping of the commonality is different from Mel’s (see Section 1); so too is Doug’s expression of it. While Mel saw the figures as made up of two horizontal lines and expressed generality in a verbal form, Doug saw the figures as recursively built by the addition of two circles diagonally arranged and expressed it dynamically through gestures and words.

In more general terms, what we observed in the classroom from the first day was that the perceptual act of noticing unfolds in a process mediated by a multi-semiotic activity (spoken words, gestures, drawings, formulas, etc.) in the course of which the object to be seen emerges progressively. This process of noticing I have termed a process of *objectification*.

The term objectification has its ancestor in the word *object*, whose origin derives from the Latin verb *obiectare*, meaning “to throw something in the way, to throw before”. The suffix –*tification* comes from the verb *facere* meaning “to do” or “to make”, so that in its etymology, objectification becomes related to those actions aimed at bringing or throwing something in front of somebody or at making something apparent –e.g. a certain aspect of a concrete object, like its colour, its size or a general mathematical property. Now, to make something apparent, students and teachers make recourse to signs and artefacts of different sorts (mathematical symbols, graphs, words, gestures, calculators and so on). These artefacts, gestures, signs and other semiotic resources used to objectify knowledge I call *semiotic means of objectification* (a detailed account can be found in Radford, 2003; 2002c).

In our previous example, Doug started making apparent a general mathematical structure –he started objectifying it. To accomplish this, Doug resorted to two semiotic means of objectification: words and gestures. In addition to highlighting the last two circles, the rhythmic repetition of gestures allowed Doug to achieve something notable: through this, Doug expressed the idea of something *general*, something that continues *further and further*, in space and in time.

I am not suggesting, though, that Doug’s six gestures and one utterance were enough to fully disclose the mathematical structure behind the pattern. Neither am I affirming that Doug was providing a direct expression of whatever term of the sequence. What I am saying is that the objectification of the general goes through various layers of awareness. To get a better grasp of the structure behind the pattern, Doug’s process of objectification had to continue. Through mediating signs, Doug continued engaging with the object of knowledge and signifying generality in more precise terms. It is obvious that the sense of generality achieved through words and gestures is not the same as the one achieved through a formula or a graph. A semiotic system provides us with specific ways to signify or to say certain things, while another semiotic system provides us with other ways of signification. The linguist Émile Benveniste referred to this situation as the principle of *nonredundancy*: “Semiotic systems”, Benveniste said, “are not ‘synonymous’; we are not able to say ‘the same thing’ with spoken words that we can with music, as they are systems with different bases.” (Benveniste in Innis, 1985, p. 235). The same distinction is true of gestures and formulas.

By the same token, Benveniste’s nonredundancy principle warns us against the common belief in translatability –the belief that e.g. a formula says *the same thing* as its graph, or that a formula says *the same thing* as the word-problem it “translates” (see e.g. Duval, 2002; Radford, 2002b). The nonredundancy principle does not mean, however, that what we intend or express in one semiotic system is completely independent from what we express in another one. The objectification of the mathematical structure behind a pattern that was mediated by words and gestures may be deepened by an activity mediated through other types of signs.

As previously described, the objectification of knowledge is a theoretical construct to account for the way in which the students engage with something in order to notice and make sense of it. By focusing on the students’ phenomenological mathematical experience, it emphasizes the subjective dimension of knowing. But this is only half of the story. Since we are sociocultural knowers, objectification takes also account of the social and cultural dimensions of knowing. The concept of knowledge objectification rests indeed on the idea that classrooms are not merely a bunch of external conditions to which the students must adapt. Classrooms are rather seen as interactive zones of mediated activities conveying scientific, ethical, aesthetical and other culturally and historically formed values that the students objectify through reflective and active participation (Radford, in press). In these activities, embedded in cultural, historical traditions, the students relate not only to the objects of knowledge (the *subject-object plane*), but also to other students through face-to-face, virtual or potential communicative actions (the *subject-subject plane* or *plane of social interaction*).

Within the previous theoretical context, our investigation of the students’ use of signs and processes of meaning production in algebra focused on a detailed study of the students’ knowledge objectification as they moved along different layers of generality and awareness.

Guided by our definition of algebraic generalization and theoretical framework, some of the research questions that we tackled were the following:

1. How do the students *grasp* the commonality in a pattern?
2. What are the mechanisms (linguistic or others) through which the students *generalize* the locally observed commonality to *all* the terms of the sequence?
3. How do they *express* generality?

In the rest of this paper, I discuss these questions focusing particularly on the work done by one of the Grade 8 (13-14 years old) and also one of the Grade 9 (14-15 years old) small-groups which are representative of most of the work done by other groups.

3. Genus Formation: Grasping and Generalizing a Local Commonality

Roughly speaking, our classroom activities were organized along the two aforementioned subject-object and subject-subject planes as follows:

The students were presented with patterns whose complexity was commensurate to the curriculum requirements. Working in small groups, the students were invited to carry out:

1. an arithmetic investigation (often conducted by continuing the pattern on the basis of some given information, as well as answering questions about specific figures such as Figure 10, Figure 25, Figure 100);
2. the expression of generalization in natural language (in the form of a message), and
3. the use of standard algebraic symbolism to express generality.

Among the patterns that we selected, there were some classical circle and toothpick patterns (such as those shown in Figure A and Figure D) and variations of increased difficulty as the students moved through their Junior and Senior High School years (see Bardini, Radford, & Sabena, 2005).

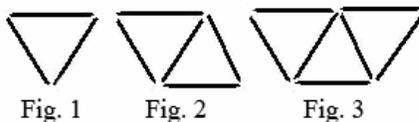


Figure D. An example of a toothpick pattern.

As indicated in Section 1, using different techniques, the students usually succeeded in answering questions about Figure 10, Figure 25, Figure 100, etc. Let us put aside the inductive, non generalizing techniques, and focus on the generalizing strategies only.

When adolescent or younger students tackle questions about “big” figures, such as Figure 25 or 100, a frequent strategy consists in noticing a recurrent relation between consecutive figures (see e.g. Castro Martínez, 1995, and Warren, 2006, respectively). This typical strategy is illustrated in the following excerpt from a Grade 8 small-group, concerning Figure 25 of the toothpick pattern:

1. Judith: The next figure has two more than ... look ... [...] [Figure] 6 is 13, 13 plus 2. You have to continue there [...].
2. Anik: Well, you can't always go plus 2, plus 2, plus 2...
3. Judith: But of course! That's Figure 7, plus 2 equals Figure 8.
4. Josh: That will take too long!

As the dialogue implies, the students noticed that the terms of the sequence increase by two. Furthermore, the dialogue provides us with a clear indication that, for the students, this common

increment applies not only to the terms that were explicitly mentioned but also to the terms that followed. One unambiguous indicator is the expression: “you have to continue there”. However, up to this point, the students did not make use of the already-noticed regularity to provide an exact value for the number of toothpicks in Figure 25. Actually, as line 4 indicates, they were aware that their procedure was unpractical. According to our definition of algebraic generalization, the students have not yet stepped into the realm of algebra. They did generalize something but are still in the realm of arithmetic. What they generalized was a local commonality observed on some figures, without being able to use this information to provide an expression of whatever term of the sequence. A generalization of this kind I will call *arithmetic generalization*.

Trying to come up with another strategy, Josh proposed a more direct procedure:

1. Josh: It’s always the next. Look! (*Then, pointing to the figures with the pencil he says*) 1 plus 2, 2 plus 3 [...]
2. Anik: So, 25 plus 26...

Line 1 shows the moment at which Josh realized that there was a different commonality linking the number of toothpicks in a figure and the sum of the ranks of two consecutive figures. The utterance “It’s *always* the next” (my emphasis) indicates Josh’s awareness that this commonality applies to all the terms. Drawing on Josh’s idea, Anik was then able to directly provide an expression for the value of Figure 25. Thus, the students here made an algebraic generalization –one that in a previous work (Radford, 2003) I have referred to as *factual generalization*.

The adjective *factual* stresses the idea that this generalization occurs within an elementary layer of generality –one in which the universe of discourse does not go beyond particular figures, like Figure 1000, Figure 1000000, and so on. This layer of generality is rather the layer of *action*: The *genus* of the sequence leads to the formation of a *schema* that operates on particular numbers (e.g. “1 plus 2, 2 plus 3”, see Line 1). Another way to say this is that in factual generalizations, *indeterminacy* –the first characteristic of algebraic thinking mentioned in the Introduction– does not reach the level of enunciation: it is *expressed in concrete actions* (see also Vergnaud’s (1996) “theorem-in-act”).

Of course, the students had pragmatic reasons to remain bounded to the factual level of generality. Factual generalization was good enough to get the answers that we asked of them. This was not to be the case when the students tackled the next question. Before going there, I want to discuss another excerpt, from a Grade 9 class, dealing with the sequence shown in Figure A.

This group was formed by three students: Jay, Mimi (sitting side by side) and Rita (sitting in front of them). Prior to the excerpt that I am going to present, the students found that the number of circles in Figures 10 and 100 was 23 and 203 respectively. They perceived the given figures as formed by two horizontal rows, generalize this commonality to the other figures of the sequence and formed a factual generalization (“11 and 12”, “101 and 102”; see Sabena, Radford, and Bardini, 2005). However, Mimi was intrigued by the fact that the digit 3 was at the end of the answers. In the excerpt which follows she tries to come up with another generalizing schema that would include the digit ‘3’ and the number of the figure:

1. Mimi: Add... Add three to the number of the figure! (*pointing to the results “23” and “203” already written on the paper*).

2. Jay: No! 101 (meaning the top row of Figure 100), 100 (meaning Figure 100) and you got that, 203.

In line 1, Mimi tried to formulate a new schema. As Jay quickly noticed, the schema is faulty (line 2). Jay’s utterance was followed by a long pause (5.2 seconds) during which the students silently looked at the figures. Jay became interested in Mimi’s idea but, like Mimi, still did not see the link in a clear way.

Trying to come up with something, while putting his pen on Figure 1 and echoing Mimi’s utterance, Jay pensively said: “Add 3”. At the same time, Mimi moved her finger to Figure 1 (close to Jay’s pencil) and said: “I mean like ... I mean like ...” (see Figure E). Right after She intervened again and said: “You know what I mean? Like... for Figure 1 (making a gesture; see Figure F, left) you will add like (making another gesture; see Figure F, right) ...”

To explore the role of digit 3, Mimi made two gestures.

The first one has an indexical-associative meaning: it indicates the first circle on the top of the first row and associates it to Figure 1 (Figure F, left). The second one digit 3 and three (Figure F, right) or pointed to the first has been noticed, i.e., bottom has remained

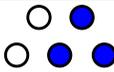
for Figure 1	you will add
	
	
<p>Figure F. Perceptual objectifying effects of word and gesture on Figure 1.</p>	



Figure E. Jay and Mimi pointing at Figure 1, trying to notice a commonality.

achieves a meaningful link between “remarkable” circles in the figure. Although Mimi has not mentioned circle on the bottom row, the circle although the first circle on the outside the realms of word and gesture, it has fallen into the realm of vision. Indeed, right after finishing her previous utterance, Mimi starts with a firm “OK!” that announces the recapitulation of what has been said and the opening up towards a deeper level of objectification, a level where all the circles of the figures will become objects of discourse, gesture and vision. She says:

Mimi: OK! It would’be like one (indexical gesture on Figure 1; see Picture 1), one (indexical gesture on Figure 1; see Picture 2), plus three (grouping gesture; see Picture 3); this (making the same set of gestures but now on Figure 2) would’be two, two, plus three; this (making the same set of gestures but now on Figure 3) would be three, three, plus three.

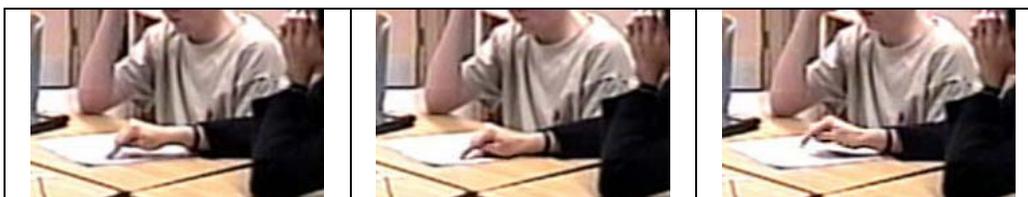


Figure G. In Pictures 1 and 2 Mimi makes an indexical gesture to indicate the first circle on the top row and the first circle on the bottom row of Figure 1; in Picture 3, she makes a “grouping gesture” to put together the last three circles of Figure 1.

Making two indexical gestures and one “grouping gesture” that surrounds the three last circles on Figure 1, Mimi rendered a specific configuration visible to herself and to her group-mates. This set of three gestures was *repeated* as she moved to Figure 2 and Figure 3. In so doing, Mimi made apparent a local commonality. Now, how does she manage to *generalize* it to all the terms of the sequence? We are here at the kernel of the generalization process. To answer this question, let us pay attention to Mimi’s semiotic means of objectification.

We have already noted the crucial objectifying role of gestures. However, Mimi’s gestures were accompanied by the *same* sentence structure (see Figure H). Through repetition and a coordination of gestures and words, Mimi generalized a locally perceived commonality to the other figures and moved from the particular to the general.

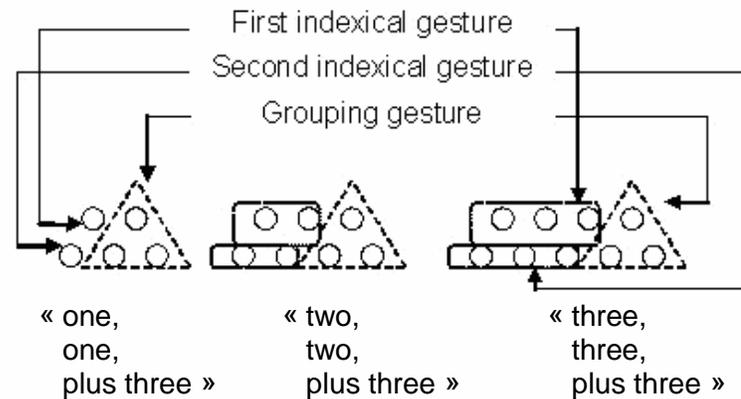


Figure H. Mimi’s objectification of a new genus of the sequence.

But in fact, in addition to gestures and words there was also *rhythm*. Rhythm was also present in Anik’s utterance quoted in the first example of this section. Rhythm creates the expectation of a forthcoming event (You, 1994) and constitutes a crucial semiotic means of objectification to make apparent the feeling of an order that goes beyond the particular figures (for a detailed discussion of rhythm see Radford, Bardini and Sabena, 2006).

Mimi’s generalization was hence forged with words, gestures and rhythm. Her generalization led to a schema through which the students were able to directly determine the number of circles in any *particular* figure. It is a factual generalization.

4. Showing versus Saying

Let us now discuss how students tackled the question concerning the expression of generality in natural language. The students were asked to write a message explaining how to find the number of toothpicks or circles in *any* figure to an imaginary student in another class of the same level. The level of generality that is required here is of course greater; for one thing, factual generalizations are no longer sufficient.

In Josh’s group, Anik suggested a first idea:

We can say, like, it’s the number of the figure, right? Like, let’s say it’s 1 there. If ... if ... OK. You add ... like, how do you say that? In order of ... (*Then, implicitly referring to Figure 2, she says*) You add it by itself, like. You do 2 plus 2, then after this, plus 1, like. You always do this, right? [...] You would do (*while she rhythmically mentions the numbers to reveal the underlying commonality, she gestures as if pointing to something*) 3

plus 3 ... plus 1, 4 plus 4 ... plus 1, 5 plus 5 ... plus 1. Do you know what I want to say?
[...] How do we say it then?

The problem, as Anik mentions, is how to express in words something general that is nonetheless easy to show through numbers and gestures. There is, in fact, a profound gap between showing and saying. The expression of the genus of the sequence (be it the first one objectified earlier by this group, based on the addition of consecutive ranks of figures, or the new one, suggested by Anik here) now has to fall in the realm of language. Indeterminacy has to be named.

After a series of unsuccessful efforts, Anik came back to their previous factual generalization:

1. Anik: Yes. Yes. OK. You add the figure plus the next figure ... No. Plus the ... [...] (*she writes as she says*) You add the first figure...
2. Josh: (*interrupting and completing Anik's utterance says*) ... [to] the second figure [...]
3. Anik: So...(*inaudible*). It's not the second figure. It's not the next figure?
4. Josh: Yes, the next one [figure].
5. Judith: Uh, yes, the next [figure] [...]
6. Anik: (*summing up the discussion*) You add the figure and the next figure.

To name indeterminacy in the message, the students transformed the expression “any figure” (as mentioned in the question) into “the figure” –a linguistic generic expression that does not designate a particular term of the sequence but whatever term you want to consider. The concrete actions on which the students' previous factual generalizations were based (“1 plus 2”, etc.) appear now as a *single* action, as an *action in abeyance*: “You add the figure and the next figure.”

The above generalization is located at a deeper layer of generality, one in which rhythm and ostensive gestures have been excluded. The students have to work here with reduced forms of expression. At the same time, to succeed at this level of generality, the students have to compensate for the reduction of semiotic resources with a *concentration of meanings* in the fewer number of signs (words) through which the generalization is now expressed. This reduction of signs and concentration of meanings constitutes a *semiotic contraction* (Radford 2002c; see also Duval 2002).

To distinguish these kinds of generalizations from factual ones, I termed them *contextual generalizations* (Radford 2003). They are contextual in that they refer to contextual, embodied objects, like “the next figure” which supposes a privileged viewpoint from where the sequence is supposedly seen, making it thereby possible to talk about *the* figure and the *next* figure.

The expression of generality beyond the level of factual generality has been investigated in the context of early algebra research. At the PME 2006 Conference, Elizabeth Warren reported a study with Grade 5 students (10 years old). Among other things, she asked the students to write in natural language the general rule for some patterns and found that between 6 and 10 students out of 27 were able to write a relationship between the position of the term and its numerical value, while between 16 and 21 students failed to do so (Warren, 2006). At the same conference, Ferdinand Rivera reported results from a research project conducted with Grade 6 students (11 years old) (Rivera, 2006). The students were presented with a slightly modified version of the sequence shown in Figure A. The terms started with one circle and increased by two circles. The students had to write a message to an imaginary Grade 6 student clearly explaining what s/he must do in order to find out how many circles there were in any given figure of the sequence.

Two answers were the following. Student 1: “You start at one and keep adding two until you get the right number of circles in all”. Student 2: “You look at the figure number and then draw the number of circles then going up you put a # less then add it all together.”

There are several interesting features in the answers. Student 2 took advantage of the geometric shape of the figures to form a *genus* of the sequence and provided a contextual generalization, the embodied dimension of which appears in the situated description of the actions as in “going up”. Student 1 formed a different genus: the common increment of two circles between figures. However, the student did not provide a direct expression for any given figure. This is hence an example of an arithmetic generalization that does not reach an algebraic character.

Let us come back to Mimi’s group. The students continued refining the factual generalization that we discussed in Section 3. Mimi said:

1. Mimi: The number of the figure like ... we’ll say that the figure is 10 (*gesture with an open hand as to indicate a row on the desk*), you’ll have ten dots (*similar gesture on the desk*) plus three (*sort of grouping gestures a bit more to the right and to the bottom, on the desk*) right? (*pause*) No...
2. Jay: (*Almost simultaneously*) No.
3. Mimi: You double the number of the figure.
4. Jay: ten plus ten (*pointing to the sheet*)
5. Mimi (*interrupting*): So it will be twenty dots plus three (*pointing to the number 23 on the sheet*). You double the number of the figure and you add three, right? So Figure 25 will be fifty...three. Right? That’s what it is [...]
6. Jay: Figure times two plus three.

The written message was the following: “The number of the figure $\times 2, + 3$. It gives you the amount of circles.”

The message is a mixture of mathematical symbols and terms in natural language. Undoubtedly, the comma is the most interesting element: it translates, in a written form, the spatial and temporal characteristics of one crucial distinctive event objectified in the course of the students’ mathematical experience, namely the distinction between the same and the different elements in the figures, as the students perceived them.

5. Writing Little while Saying a Lot

In Josh’s group, expressing the generalization through alphanumeric symbols –what I have called a *symbolic generalization* (Radford, 2003)– was a complex process during the course of which the students had to decide about the meaning of letters. One particular problem was to decide how to say “the next figure”. The following excerpt illustrates some of the difficulties:

1. That would be like $n + a$ or something else, $n + n$ or something else.
2. Anik: Well [no] because “a” could be any figure [...] You can’t add your 9 plus your ... like ... [...] You know, whatever you want it has to be your next [figure].

When the students reached an impasse, the teacher intervened: “If the figure I have here is ‘n’, which one comes next?” Thinking of the letter in the alphabet that comes after n , Josh replied: “o”. In the end they ended up with the following formula: “ $(n+1) + n$ ”.

The formula in Jay's group was as follows: " nx^2+3 ". Formed out of a commonality noticed through a complex coordination of hands in space, rhythm, nouns, deictics and adverbs, the formula reached here an extremely concise expression. The "space" to be occupied by each one of its five signs (i.e. "n", "x", "2", "+", and "3") was progressively prepared by the students' previous joint mathematical experience. Thus, the symbolic letter n is the "semiotic contraction" of the "number of the figure" that has been so often quoted before, either directly or by means of examples. In fact, the whole formula is the crystallization of a semiotic process endowed with its situated history. It is a history in which each sign acquired a distinctive meaning and which may explain why the students do not simplify the formula into the more standard expression: $2n + 3$. The formula still hangs behind the remnants of the *narrative* side of algebra (Radford, 2002b), where signs play the role of narrating a story and where the formula has not yet reached the autonomy of a detached symbolic artifact. The letters of which a formula is made up play indeed the role of *indexes* pointing to words of the students' contextual and factual generalizations.

Obviously, some students' formulas do not correspond to the standard algebraic syntax. Thus, dealing with the sequence shown in Figure A, Samantha, one Grade 8 student, managed to produce a contextual generalization: "You must add 1 more than the figure for the top and 2 more on the bottom." Her formula was: " $(n+1)+2=$ ". Now, despite its inaccurate algebraic syntax, the formula was not written at random. A closer look at the formula indeed suggests that the formula does have a meaning. The formula was built following a syntax based on the criterion of *juxtaposition of signs*. It is a sentence structured in the manner of a *narrative* where signs become encoded as *key terms* (much as ideograms did in the written language used in Mesopotamia ca 3500 BC –where, e.g., the drawing of a foot after the drawing of a mountain in a clay tablet could mean a long walk). The formula is recounting us Samantha's mathematical experience with the general. The composed term " $n+1$ " is telling us that, to determine the number of circles on the top row, we have to add 1 more (circle) than the (number of the) figure, and that once we have finished doing this (something scrupulously indicated by the brackets), we still have to add two (circles) to the bottom row. Now, by adding these results, we may be in a position to find the total number of circles in the figure. The inaccuracy of algebraic syntax cannot be imputed to Samantha's misunderstanding of the problem: she succeeded in finding the number of circles in Figure 10 and Figure 100. Had we asked her questions about "bigger" figures, like Figure 1000000, she would have provided the right answers. The problem lies elsewhere. It lies in the students' understanding of a cultural mathematical practice based on a specific use of signs.

6. Synthesis and Concluding Remarks

Noticing a commonality in a few particular terms of a sequence is by no means the result of a contemplative act. As Kant put it:

I see a fir, a willow, and a linden. In firstly comparing these objects, I notice that they are different from one another in respect of trunk, branches, leaves, the like; further, however, I reflect only on what they have in common ... and abstract from their size, shape, and so forth; thus I gain a concept of tree. (Kant, 1974, p. 100).

Our ability to notice differences in things is one of our basic cognitive components. Without it, we would be unable to sort the amazing amount of sensorial stimuli that we receive from the exterior and the world in front of us would be reduced to an amorphous visual, tactile and aural mass. Naturally, as many of Kant's commentators have pointed out, things are a good deal more

complicated than Kant himself suggested. Noticing the differences and similarities that lead to the *genus* of a pattern in our case or to the genus of a tree in Kant's own example, occur in social activities subsumed in cultural traditions conveying ideas about the *same* and the *different* and about how these differences may be *reflected* and *abstracted*. This is why some cultures make finer or different categorizations of plants and colors than others. We certainly notice differences and similarities—not through neutral tactile, aural, visual and other sense impressions—but through our historically and culturally species-evolved senses (Gibson, 1966; Wartofsky, 1979). So, instead of a contemplative and obvious act, to notice something –anything, trivial though it may be, like the *circles* in a pattern– is already a complex cultural-cognitive process.

Now, we do not remain confined to what we materially see –perception, it is true, is always the perception of particulars. We go beyond the realm of particulars by noticing something else –something *general, conceptual*– and by trying to make sense of it. I referred to this process of concept-noticing and sense-making as a process of objectification.

The whole idea of objectification is embedded in an ontology according to which the concepts or objects of knowledge are made up of *layers of generality*. The epistemological counterpart to this ontological premise asserts that our knowledge of a certain conceptual object is concurrent with the layers of generality in which we can deal with the object. Because each one of these objects' layers is general, they cannot be fully grasped in the realm of the particular. The diaphanous or insubstantial general can only come into being through signs. This is why to objectify something is to make it come into the world of (re)presentation, i.e. to appear within a semiotic process.

In this line of thought, I have suggested distinguishing between the diverse strategies that the students use when they deal with the generalization of patterns. Patterning activity has been justly considered as one of the prominent routes for introducing students to algebra. However, not all patterning activity leads there. This is the case of inductive procedures based on rule formation by trial and error and other guessing strategies. These procedures do not lead to algebra because algebra is certainly neither about guessing nor about just using signs. It is rather about using signs *to think in a distinctive way*. As far as patterns are concerned, algebra is about generalizing. Now, as Kant's example intimates, to talk about generalizing is to talk about two things: (1) that which is generalized (the object of generalization), and (2) the generalized object. Drawing on Kieran (1989), Love (1986) and Mason (1996), I have suggested that the process that goes from one to the other includes two interrelated components. The first one is noticing a commonality in some given particular terms. The second one is to form a general concept –a *genus*– by generalizing the noticed commonality to all the terms of the sequence. In order for a generalization of patterns to be called algebraic, I have suggested a third component: that the genus or generalized object crystallize itself into a *schema*, i.e. a rule providing one with an expression of whatever term of the sequence (arithmetic generalizations would be those which fail to meet the third component). Next, I have discussed three layers of algebraic generality and the corresponding modes of expression: factual, contextual and symbolic (see Table 1).

Naïve Induction	Generalization		
Guessing (Trial and Error)	Arithmetic	Algebraic	
		Factual	Contextual Symbolic

Table 1. Students' strategies for dealing with pattern activities and the subdivision of algebraic generalizations in accordance with their level of generality.

These layers of generality are characterized by the semiotic means of objectification to which the students resort in order to accomplish their generalizations. In factual generality, indeterminacy remains unnamed; generality rests on *actions* performed on numbers; *actions* are made up here of words, gestures and perceptual activity. In the contextual and symbolic layers of generality, the indeterminate is made linguistically explicit: it is *named*. While in contextual generality the general objects are named through an embodied and situated description of them (e.g. “the next figure”, “the top row”, etc.), in symbolic generality the general objects and the operations made with them are expressed in the alphanumeric semiotic system of algebra.

Factual generality provides the raw material that, through successive *semiotic contractions*, the students will later transform into higher forms of algebraic generality. The issue here is not just to say the same thing in a different code. It is rather about gaining access to deeper forms of consciousness. It is in this respect that the genetic link between layers of generality is most revealing. For instance, we saw the tremendous cognitive importance of words, gestures and perceptual activity in factual generality (as expressed in “1 plus 2”, etc.) and their important objectifying effects: they prepare the space where the designation of objects may occur later and where the students’ consciousness of indeterminacy may reach a deeper layer of objectification.

In this context, an important question to ask is the following: Why did the students gesture? Why did they not limit themselves to talking? Gestures helped the students to refine their awareness of the general. These gestures stood for the rows that *could not be seen*. Gestures helped the students to *visualize* (Presmeg, 2006) and hereby came to fill the gap left by impossible direct perception. Generally speaking, gestures do not merely carry out intentions or information; they are key elements of the process of knowledge objectification (Radford, 2005b)³.

From an educational perspective, it is important to bear in mind that each one of the layers of generality presents its own challenges. As we saw in the classroom examples discussed earlier, in factual and contextual generalizations, the students often talk about “the figure” instead of “the number of the figure”; because of the embodied and metonymic mode of designation of objects, the students’ generalizations often carry some ambiguities. In symbolic generalizations, the students’ formulas often tend to simply narrate events and remain attached to the context. The understanding and proper use of algebraic symbolism entails the attainment of a disembodied cultural way of using signs and signifying through them. The disembodiment of meaning of symbolic generalizations I am talking about should nevertheless not be understood as the decline or elimination of the individual, but as a new way of engaging with, and reflecting about, the general and the particular (see Radford, 2006, p. 60; see also Roth, 2006). This attainment, I want to suggest, can only be possible through a transformation of the way in which letters signify in a formula. In addition to their indexical mode of signification, letters have to acquire a symbolic mode as well. In Peirce’s terminology, letters have to become genuine symbols. The didactic situations that may promote the transformation of the index into symbol in the students’ formulas have still to be investigated further (see Barallobres, 2005). When we invited our students to simplify formulas, some progress in the direction from index to symbol was observed, even with the youngest. Thus, several groups of Grade 8 students went from “ $r+r+r+1$ ” to “ $rx3+1$ ”. However, examples such as these are not enough to provide us with a clear idea of the genetic path that goes from one mode of signification to the other. My conjecture at this point is that this path is paved with subtle qualitative changes where indexicality is progressively put in the background and the letters acquire a relational meaning (see Radford and Puig, in press).

Be this as it may, I hardly believe that the didactic situations susceptible to leading our students to deeper layers of symbolic or other forms of generality can be reduced to the choice of fortuitously good mathematical problems. Powerful though it may be, the plane *subject-object* is not, epistemologically speaking, strong enough. The plane of *social interaction* must be included. The students have to learn to see the objects of knowledge from others' (teachers and students) perspectives. This is why, in the classroom, we often organized an exchange of ideas and solutions and the discussion of them between groups, followed by general class discussions (Radford & Demers, 2004). The idea, however, is not merely to 'share' solutions in order to *catalyze* the attainment of deeper layers of generality. It is rather that the objectification of knowledge presupposes the encounter with an object whose appearance in our consciousness is only possible through contrasts. Our awareness and understanding of an object of knowledge is only possible through the encounter with other individuals' understanding of it (Bakhtin, 1990; Hegel, 1977; Vygotsky, 1962). In this encounter, our understanding becomes entangled with the understandings of others and the historical intelligence embodied in cultural artifacts (e.g. language, writing) that we use to make our experience of the world possible in the first place.

Acknowledgment

This article is a result of a research program funded by The Social Sciences and Humanities Research Council of Canada (SSHRC/CRSH). I wish to thank Raymond Duval and Luis Puig for their insightful comments on a draft of this paper.

EndNotes

(1) The concept of induction has been the object of a vast number of investigations in epistemology and in education; see e.g. Peirce in Hoopes, 1991, pp. 59-61; Polya, 1945, pp. 114-121; Poincaré, 1968 p. 32 ff.). In what follows, to simplify the text, I will use induction to refer to the students' *naïve* induction described above.

(2) Because of the students' strong tendency to use inductive procedures instead of generalizing ones, we proposed some patterns with decimal numbers. One of those patterns was the following: 0.82, 1.13, 1.44, 1.75, 2.06, ... Here, the possible values of a and b in the rule " $an+b$ " (or " $nxa+b$ " as the students would write) increase considerably making trial and error a heuristic which is no longer viable. As one of the students commented after failing at several trial and error guesses, "I got more numbers in my head than ever".

(3). Currently, there is an intense interest in gestures in general, as well as in science and mathematics education. Some recent work includes Arzarello and Edwards, 2005; Goldin-Meadow, 2003; Kendon, 2004; Kita, 2003; McNeill, 2000; Robutti, in press; Roth, 2001.

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